A NOTE ON LOG CANONICAL THRESHOLDS

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Abstract. We prove that the largest accumulation point of the set $\mathcal{T}_3$ of all three-dimensional log canonical thresholds $c(X, F)$ is 5/6.

1. Introduction

Let (X, Ω) be a log variety and let F be an effective non-zero Weil $\mathbb{Q}$-Cartier divisor on X. Assume that (X, Ω) has at worst log canonical singularities. *The log canonical threshold* of F with respect to (X, Ω) is defined by

$$c(X, \Omega, F) = \sup \{c \mid (X, \Omega + cF) \text{ is log canonical}\}.$$

It is known that $c(X, \Omega, F)$ is a rational number from the interval $[0, 1]$ (see [3]). We frequently write $c(X, F)$ instead of $c(X, 0, F)$.

For each $d \in \mathbb{N}$ define the set $\mathcal{T}_d \subset [0, 1]$ by

$$\mathcal{T}_d := \left\{ c(X, F) \;\middle|\; \begin{array}{c} \dim X = d,\ X \text{ has only log canonical singularities} \\ \text{and } F \text{ is an effective non-zero Weil } \mathbb{Q}\text{-Cartier divisor} \end{array} \right\}.$$

The structure of $\mathcal{T}_d$ is interesting for applications to the problem of termination some inductive procedures appearing in the Minimal Model Program [10], [5]. The interest in log canonical thresholds was also inspired in connection with the complex singular index and Bernstein-Sato polynomials (see [3]).

Conjecture 1.1 ([10])**.** *$\mathcal{T}_d$ satisfies the ascending chain condition, i.e. any increasing chain of elements terminates.*

The set $\mathcal{T}_2$ is completely described (see [7]). Concerning $\mathcal{T}_3$ it is known the following:

(i) Conjecture 1.1 holds true for $\mathcal{T}_3$ [1], [5, Ch. 18];
(ii) $\mathcal{T}_3 \cap (41/42, 1) = \varnothing$ [4];
(iii) $\mathcal{T}_3 \cap [6/7, 1]$ is finite [9].

Actually, the structure of $\mathcal{T}_d$ is rather complicated: it has a lot of accumulation points [3, 8.21]. However adopting Conjecture 1.1 we see that $\mathcal{T}_d$ is discrete near 1.

This work was partially supported by the grant INTAS-OPEN-97-2072.

Our main result is the following theorem which generalizes the result of [9].

Theorem 1.2. *The largest accumulation value of $\mathcal{T}_3$ is* $5/6$.

Remark 1.3. (i) The two-dimensional analog of our theorem easily follows from the description of $\mathcal{T}_2$ ([7]): the largest accumulation value of $\mathcal{T}_2$ is $1/2$.

(ii) T. Kuwata described the set of all values $c(\mathbb{C}^3, F)$ in the interval $[5/6, 1]$, where F is a hypersurface in $\mathbb{C}^3$. His proof is done by studying the local equation of F. Our proof uses quite different method and based on Alexeev's result [2].

The essential part of the proof is to show the finitedness of $\mathcal{T}_3 \cap [5/6 + \epsilon, 1]$ for any $\epsilon > 0$. The easy example below shows that $5/6$ is an accumulation point of $\mathcal{T}_3$.

Example 1.4. Let $X = \mathbb{C}^3$ and let F_r be the hypersurface given by $x^2 + y^3 + z^r$, $r \geq 7$. This singularity is quasihomogeneous. By [3, 8.14] we have $c(\mathbb{C}^3, F_r) = 5/6 + 1/r$. Thus $\lim_{r \to \infty} c(\mathbb{C}^3, F_r) = 5/6$.

Acknowledgments. This work was completed during my stay at Max-Planck-Institut für Mathematik. I would like to thank MPIM for hospitality and support. I am grateful to Dr. O. Fujino for pointing out several inaccuracies in the first draft.

2. Preliminary results

All varieties are assumed to be algebraic varieties defined over the field $\mathbb{C}$. A *log variety* (or a *log pair*) (X, D) is a normal quasiprojective variety X equipped with a *boundary*, a $\mathbb{Q}$-divisor $D = \sum d_i D_i$ such that $0 \leq d_i \leq 1$ for all i. We use terminology, definitions and abbreviations of the Minimal Model Program [5].

Proposition-Definition 2.1 ([10, §3], [5, Ch. 16])**.** Let $(X, S + B)$ be a log variety, where $S = \lfloor S + B \rfloor \neq 0$ and divisors S, B have no common components. Assume that $K_X + S$ is lc in codimension two. Then there is a naturally defined effective $\mathbb{Q}$-divisor $\mathrm{Diff}_S(B)$ on S called the *different* of B such that

$$K_S + \mathrm{Diff}_S(B) \sim_{\mathbb{Q}} (K_X + S + B)|_S.$$

2.2. Let Φ be a subset of $\mathbb{Q}$. For a $\mathbb{Q}$-divisor $D = \sum d_i D_i$, we write $D \in \Phi$ if $d_i \in \Phi$ for all i. Define the following sets

$$\Phi_{\mathbf{sm}} := \big\{1 - 1/m \mid m \in \mathbb{N} \cup \{\infty\}\big\},$$
$$\Phi_{\mathbf{sm}}^{\alpha} := \Phi_{\mathbf{sm}} \cup [\alpha, 1], \quad \text{for } \alpha \in [0, 1].$$

We distinguish them because they are closed under some important operations (see e.g. Corollary 2.5 below). Usually the numbers from $\Phi_{\mathbf{sm}}$ are called *standard*.

Proposition 2.3 ([10, Prop. 3.9]). *Let (X,S) be a d-dimensional plt log variety, where S is integral. Let $W \subset S$ be an irreducible subvariety of codimension 1. Then near the general point $P \in W$ there is an analytic isomorphism*

$$(X,S,W) \simeq \Big((\mathbb{C}^d, \{x_1=0\}, \{x_1=x_2=0\})/\mathbb{Z}_m(1,q,0\dots,0)\Big), \qquad \text{where} \quad m,\ q\in\mathbb{N},\ \gcd(m,q)=1. \tag{2.1}$$

Corollary 2.4 ([10, 3.10, 3.11]). *Let $(X,S+B)$ be a log variety, where $S := \lfloor S+B \rfloor$ and divisors S, B have no common components. Assume that (X,S) is plt. Let $W \subset S$ be an irreducible subvariety of codimension 1. If $B = \sum b_i B_i$, then the coefficient of $\mathrm{Diff}_S(B)$ along W is equal to*

$$1 - \frac{1}{m} + \sum_{B_i \supset W} \frac{n_i b_i}{m}, \tag{2.2}$$

where m is such as in (2.1) and $n_i \in \mathbb{N}$. Moreover, if $(X,S+B)$ is plt and $B \in [1/2,1]$, then there is at most one component B_i of B containing W and $n_i = 1$.

Corollary 2.5 ([10, 3.11, 4.2]). *Let $(X,S+B)$ be a log variety, where $S := \lfloor S+B \rfloor$ and divisors S, B have no common components. Assume that (X,S) is plt and $(X,S+B)$ is plt. Take $\alpha \in [0,1]$. If $B \in \Phi_{\mathbf{sm}}^{\alpha}$, then $\mathrm{Diff}_S(B) \in \Phi_{\mathbf{sm}}^{\alpha}$.*

Proposition-Definition 2.6 ([8]). Let (X,D) be a log variety such that (X,D) is lc but not plt, X is klt and $\mathbb{Q}$-factorial. Assume the log MMP in dimension $\dim(X)$. Then there exists a blow-up $f\colon Y \to X$ such that

- (i) the exceptional set of f contains an unique prime divisor S;
- (ii) $K_Y + D_Y = f^*(K_X+D)$ is lc, where D_Y is the proper transform of D;
- (iii) $K_Y + S + (1-\varepsilon)D_Y$ is plt and anti-ample over X for any $\varepsilon > 0$;
- (iv) Y is $\mathbb{Q}$-factorial and $\rho(Y/X) = 1$.

Such a blow-up we call an *inductive blow-up* of (X,D).

3. Lemmas

Lemma 3.1. *Let Λ be a boundary on $\mathbb{P}^1$ such that $\Lambda \in \Phi_{\mathbf{sm}}^{5/6}$ and $K_{\mathbb{P}^1} + \Lambda \equiv 0$. Then $\Lambda \in \Phi_{\mathbf{sm}} \cap [0, 5/6] \cup \{1\}$.*

Proof. Write $\Lambda = \sum \lambda_i \Lambda_i$. Then $\lambda_i \in \Phi_{\mathbf{sm}}^{5/6}$ and $\sum \lambda_i = 2$. If $\lfloor \Lambda \rfloor \neq 0$, then there are only two possibilities: $\lambda_1 = \lambda_2 = 1$ and $\lambda_1 = 2\lambda_2 = 2\lambda_3 = 1$. Otherwise $\lambda_i < 1$ and easy computations give us $\lambda_i \leq 5/6$, so $\lambda \in \Phi_{\mathbf{sm}}$. □

Lemma 3.2. *Let $(S, \Delta = \sum \delta_i \Delta_i)$ be a lc log surface such that $\delta_i \in \Phi_{\mathbf{sm}}^{5/6}$ and let C be an effective Weil divisor on S. Then either $c(S, \Delta, C) \leq 5/6$ or $c(S, \Delta, C) = 1$.*

Proof. Put $c := c(S, \Delta, C)$. Assume that $5/6 < c < 1$. By [3, 8.5] there is an exceptional divisor E such that $a(E, \Delta + cC) = -1$ and $a(E, \Delta) > -1$. Put $P := \mathrm{Center}(E)$. Regard S as a germ near P.

Let $\varphi \colon \widetilde{S} \to S$ be an inductive blowup of $(S, \Delta + cC)$. Write

$$K_{\widetilde{S}} + \widetilde{\Delta} + c\widetilde{C} + \widetilde{E} = \varphi^*(K_S + \Delta + cC),$$

where $\widetilde{E}$ is the exceptional divisor, $\widetilde{C}$ and $\widetilde{\Delta}$ are proper transforms of C and Δ, respectively. By Corollary 2.5, $\mathrm{Diff}_{\widetilde{E}}(\widetilde{\Delta} + c\widetilde{C}) \in \Phi_{\mathbf{sm}}^{5/6}$. On the other hand, $K_{\widetilde{E}} + \mathrm{Diff}_{\widetilde{E}}(\widetilde{\Delta} + c\widetilde{C}) \equiv 0$. By Lemma 3.1, $\mathrm{Diff}_{\widetilde{E}}(\widetilde{\Delta} + c\widetilde{C}) \in [0, 5/6]$. Clearly, $\widetilde{E} \cap \widetilde{C} \neq \varnothing$. Applying Corollary 2.2 to our situation we obtain $1 - 1/m + c/m \leq 5/6$ for some $m \in \mathbb{N}$. This yields $c \leq 5/6$, a contradiction. □

Lemma 3.3 (cf. [11])**.** *Let $(S \ni o, \Lambda = \lambda_1 \Lambda_1 + \lambda_2 \Lambda_2)$ be a log surface germ such that $\lambda_1, \lambda_2 \geq 5/6$. Assume that $\mathrm{discr}(S, \Lambda) \geq -5/6$ at o. Then $\lambda_1 + \lambda_2 \leq 11/6$.*

Proof. By Lemma 3.2, $K_S + \Lambda_1 + \Lambda_2$ is lc at o. In this situation there is an analytic isomorphism (cf. Proposition 2.3)

$$(S, \Lambda, o) \simeq (\mathbb{C}^2, \{xy = 0\}, 0)/\mathbb{Z}_m(1, q),$$

where $m \in \mathbb{N}$ and $\gcd(m, q) = 1$. Take q so that $1 \leq q < m$ and consider the weighted blow up with weights $\frac{1}{m}(1, q)$. We get the exceptional divisor E with discrepancy

$$-\frac{5}{6} \leq a(E, \Lambda) = -1 + \frac{1+q}{m} - \frac{\lambda_1}{m} - \frac{q\lambda_2}{m}.$$

Thus

$$0 \leq 1 + q - \lambda_1 - q\lambda_2 - \frac{m}{6} \leq 1 + q - \frac{5}{6}(1+q) - \frac{m}{6} = \frac{1+q-m}{6}.$$

If $m \geq 2$, this gives as $q = m - 1$ and equalities $\lambda_1 = \lambda_2 = 5/6$. In the case $m = 1$, $q = 1$ we have $0 \leq 2 - \lambda_1 - \lambda_2 - 1/6$, i.e. $\lambda_1 + \lambda_2 \leq 2 - 1/6$. $\square$

4. Proof of the main theorem

In this section we prove Theorem 1.2. First we reduce the problem to the case when X is $\mathbb{Q}$-factorial and has only log terminal singularities. These arguments are quite standard, so the reader can skip them.

Lemma 4.1. *Let (X, Ω) be a d-dimensional lc log variety such that $\Omega \in \Phi_{\mathbf{sm}}$ and let F be an effective Weil $\mathbb{Q}$-Cartier divisor on X. Assume that the log MMP in dimension d holds. Then there is a $\mathbb{Q}$-factorial d-dimensional klt variety X' and an effective Weil $\mathbb{Q}$-Cartier divisor F' on X' such that $c(X, \Omega, F) = c(X', F')$.*

Proof. We prove our lemma by induction on d. Put $c := c(X, \Omega, F)$. Clearly, we may assume that $0 < c < 1$. Consider minimal dlt $\mathbb{Q}$-factorial modification $g\colon (\tilde{X}, \tilde{\Omega}) \to (X, \Omega)$ (see [5, 17.10]). By definition, this is a birational morphism $g\colon \tilde{X} \to X$ such that $\tilde{X}$ is $\mathbb{Q}$-factorial and

$$K_{\tilde{X}} + \tilde{\Omega} + \sum E_i = g^*(K_X + \Omega)$$

is dlt, where $\tilde{\Omega}$ is the proper transform of Ω and the E_i are prime exceptional divisors (if (X, Ω) is dlt, one can take $\sum E_i = 0$). Since $c > 0$ and because $a(E_i, \Omega) = -1$, F cannot contain $g(E_i)$. Therefore the proper transform of F coincides with its pull-back g^*F. Replace (X, Ω, F) with $(\tilde{X}, \tilde{\Omega}, g^*F)$. From now on we may assume that (X, Ω) is dlt and X is $\mathbb{Q}$-factorial. There is an exceptional divisor E such that $a(E, \Omega + cF) = -1$ and $a(E, \Omega) > -1$. Regard X as a germ near a point $P \in \operatorname{Center}(E)$.

Assume that $\lfloor \Omega \rfloor \neq 0$. Let S be a component of $\lfloor \Omega \rfloor$ (passing through P). Then $(S, \operatorname{Diff}_S(\Omega - S))$ is lc [5, 17.7] and $\operatorname{Diff}_S(\Omega - S) \in \Phi_{\mathbf{sm}}$ (see Corollary 2.5). Then it is easy to see that $c(X, \Omega, F) = c(S, \operatorname{Diff}_S(\Omega - S), F|_S)$. Taking into account $\mathcal{T}_{d-1} \subset \mathcal{T}_d$ (see [3, 8.21]), we get our assertion.

Now consider the case $\lfloor \Omega \rfloor = 0$. Then (X, Ω) is klt. Since X is a germ near P, $n(K_X + \Omega) \sim 0$ for some $n \in \mathbb{N}$. Take n to be minimal with this property. Then the isomorphism $\mathcal{O}_X(n(K_X + \Omega)) \simeq \mathcal{O}_X$ defines an $\mathcal{O}_X$-algebra structure on $\sum_{i=0}^{n-1} \mathcal{O}_X(\lfloor -iK_X - i\Omega \rfloor)$ this gives us a cyclic $\mathbb{Z}_n$-cover

$$\varphi\colon X' := \operatorname{Spec}\left(\sum_{i=0}^{n-1} \mathcal{O}_X\left(\lfloor -iK_X - i\Omega \rfloor\right)\right) \longrightarrow X.$$

The ramification divisor of φ is Ω. Hence $\varphi^*(K_X+\Omega)=K_{X'}$ and X' has only log terminal singularities [5, 20.3]. Put $F':=\varphi^*F$. Then $c(X,\Omega,F)=c(X',F')$ (see [3, 8.12]). Replacing X' with its $\mathbb{Q}$-factorialization we get the desired log pair. □

4.2. **Notation.** Let X be a three-dimensional $\mathbb{Q}$-factorial normal variety with only log terminal singularities and let F be an effective Weil $\mathbb{Q}$-Cartier divisor on X. Put $c:=c(F,X)$. Let $f\colon Y\to X$ be an inductive blowup of the pair (X,cF). Write $f^*(K_X+cF)=K_Y+cF_Y+S$, where F_Y is the proper transform of F on Y and S is the exceptional divisor. Let $\Theta:=\mathrm{Diff}_S(cF_Y)$ and $\Theta=\sum\vartheta_i\Theta_i$.

4.3. **Main assumption.** Fix $\epsilon>0$ and assume that $1>c>5/6+\epsilon$. We prove that there are only a finite number of possibilities for such c.

Lemma 4.4. *$f(S)$ is a point.*

Proof. Otherwise $f(S)$ is a curve and the pair (X,cF) is lc but not klt along $f(S)$. Taking a general hyperplane section we derive a contradiction with Lemma 3.2. □

Lemma 4.5. *$(Y,S+cF_Y)$ is plt.*

Proof. Assume the converse. Then there is an exceptional divisor E such that $a(E,S+cF_Y)=-1$. Since (Y,S) is plt, $\mathrm{Center}(E)\subset E\cap F_Y$.

If $\mathrm{Center}(E)$ is a curve, then $(Y,S+cF_Y)$ is lc but not klt along $\mathrm{Center}(E)$. As in the proof of Lemma 4.4 we derive a contradiction. Thus we may assume that $(Y,S+cF_Y)$ is plt in codimension two. By Adjunction [5, Th. 17.6] this implies that $\lfloor\Theta\rfloor=0$.

Hence $\mathrm{Center}(E)$ is a point. Again by Adjunction (S,Θ) is lc but not klt near $\mathrm{Center}(E)$. As above, we have a contradiction with Lemma 3.2. □

Corollary 4.6. *(S,Θ) is klt.*

4.7. Now we are going to construct a "good" birational model $(\bar S,\bar\Theta)$ of (S,Θ). The construction is similar to that in [11]. Assumption 4.3 gives us that $\Theta\in\Phi_{\mathbf{sm}}^{5/6}$. If $\mathrm{discr}(S,\Theta)\geq-5/6$ and $\rho(S)=1$, we put $(\bar S,\bar\Theta)=(S,\Theta)$.

From now on we assume either $\mathrm{discr}(S,\Theta)<-5/6$ or $\rho(S)>1$, Since (S,Θ) is klt, there is only a finite set $\mathcal{E}$ of divisors E with $a(E,\Theta)<-5/6$ [5, 2.12.2]. Let $\mu\colon\widetilde S\to S$ be the blow-up of all divisors $E\in\mathcal{E}$ (see [5, Th. 17.10]) and let $\widetilde\Theta$ be the crepant pull-back:

$$K_{\widetilde S}+\widetilde\Theta=\mu^*(K_S+\Theta),\quad \mu_*\widetilde\Theta=\Theta.$$

Then $\operatorname{discr}(\widetilde{S}, \widetilde{\Theta}) \geq -5/6$ and again we have $\widetilde{\Theta} \in \Phi_{\mathbf{sm}}^{5/6}$. Write $\widetilde{\Theta} = \sum \vartheta_i \widetilde{\Theta}_i$ and consider the boundary $\widetilde{\Xi}$ with $\operatorname{Supp}(\widetilde{\Xi}) = \operatorname{Supp}(\widetilde{\Theta})$:

$$\widetilde{\Xi} := \sum \xi_i \widetilde{\Theta}_i, \qquad \xi_i = \begin{cases} 1 & \text{if } \vartheta_i > 5/6, \\ \vartheta_i & \text{otherwise.} \end{cases}$$

For sufficiently small positive α, the $\mathbb{Q}$-divisor $\widetilde{\Theta} - \alpha(\widetilde{\Xi} - \widetilde{\Theta})$ is a boundary. It is clear that

$$K_{\widetilde{S}} + \widetilde{\Theta} - \alpha(\widetilde{\Xi} - \widetilde{\Theta}) \equiv -\alpha(\widetilde{\Xi} - \widetilde{\Theta})$$

cannot be nef. By our assumption, $\rho(\widetilde{S}) > 1$. Note also that $(\widetilde{S}, \widetilde{\Xi})$ is lc (see Lemma 3.2). Run $K_{\widetilde{S}} + \widetilde{\Theta} - \alpha(\widetilde{\Xi} - \widetilde{\Theta})$-MMP. On each step we contract an extremal ray R such that

$$(K_{\widetilde{S}} + \widetilde{\Xi}) \cdot R = (\widetilde{\Xi} - \widetilde{\Theta}) \cdot R > 0.$$

Consider such a contraction $\varphi \colon \widetilde{S} \to S^\sharp$.

4.8. Assume that $\dim S^\sharp = 1$ and let C be a general fiber. Since $(\widetilde{\Xi} - \widetilde{\Theta}) \cdot C > 0$, there is a component $\widetilde{\Theta}_i$ of $\widetilde{\Theta}$ with coefficient $\vartheta_i > 5/6$ meeting C. Hence $\operatorname{Diff}_C(\widetilde{\Theta})$ also has a component with coefficient $> 5/6$. By Adjunction $K_C + \operatorname{Diff}_C(\widetilde{\Theta})$ is klt. On the other hand,

$$K_C + \operatorname{Diff}_C(\widetilde{\Theta}) \equiv 0 \quad \text{and} \quad \operatorname{Diff}_C(\widetilde{\Theta}) \in \Phi_{\mathbf{sm}}^{5/6}$$

(see Corollary 2.5). This contradicts Lemma 3.1.

Thus, φ is birational.

4.9. We claim that φ cannot contract a component of $\lfloor \widetilde{\Xi} \rfloor$. Indeed, assume that φ contracts a curve $C \subset \lfloor \widetilde{\Xi} \rfloor$. Take $\widetilde{\Theta}' := \widetilde{\Theta} + \alpha C$ so that $\lfloor \widetilde{\Theta}' \rfloor = C$ and $\widetilde{\Theta}' \leq \widetilde{\Xi}$. Since $C^2 < 0$, we have $(K_{\widetilde{S}} + \widetilde{\Theta}') \cdot C < 0$. Therefore

$$\left(K_{\widetilde{S}} + \widetilde{\Theta}' + \beta\big(\widetilde{\Xi} - \widetilde{\Theta}'\big)\right) \cdot C = 0$$

for some $0 < \beta < 1$. Put $\widetilde{\Theta}'' := \widetilde{\Theta}' + \beta(\widetilde{\Xi} - \widetilde{\Theta}')$. Then $\widetilde{\Theta}'' \leq \widetilde{\Xi}$, so $(\widetilde{S}, \widetilde{\Theta}'')$ is lc. Moreover $\widetilde{\Theta}'' \in \Phi_{\mathbf{sm}}^{5/6}$. Since $(\widetilde{\Xi} - \widetilde{\Theta}'') \cdot C > 0$, there is a component of $\widetilde{\Xi} - \widetilde{\Theta}''$ meeting C. By Lemma 3.2, $(\widetilde{S}, \widetilde{\Theta}'')$ is plt near $C \cap \operatorname{Supp}(\widetilde{\Xi} - \widetilde{\Theta}'')$. As in 4.8 we derive a contradiction by Lemma 3.1.

Put $\Xi^\sharp := \varphi_* \widetilde{\Xi}$ and $\Theta^\sharp := \varphi_* \widetilde{\Theta}$. By [5, 2.28],

$$\operatorname{discr}(S^\sharp, \Theta^\sharp) = \operatorname{discr}(\widetilde{S}, \widetilde{\Theta}) \geq -5/6.$$

Thus all the assumptions hold for $(S^\sharp, \Theta^\sharp)$. Again

$$K_{S^\sharp} + \Theta^\sharp - \alpha(\Xi^\sharp - \Theta^\sharp) \equiv -\alpha(\Xi^\sharp - \Theta^\sharp)$$

cannot be nef.

Continuing the process we get a new pair $(\bar{S}, \bar{\Theta})$ such that

$\rho(\bar{S}) = 1$, $\bar{\Theta} \in \Phi_{\mathbf{sm}}^{5/6}$, $(\bar{S}, \bar{\Theta})$ is klt, $K_{\bar{S}} + \bar{\Theta} \equiv 0$, and $\operatorname{discr}(\bar{S}, \bar{\Theta}) \geq -5/6$.

Note that all our birational modifications are $(K + \Theta)$-crepant. Hence

$$\operatorname{totaldiscr}(S, \Theta) = \operatorname{totaldiscr}(\bar{S}, \bar{\Theta}) = \operatorname{totaldiscr}(\widetilde{S}, \widetilde{\Theta})$$

(see [3, 3.10]). Consider the decomposition $\Theta = \Theta^a + \Theta^b$, where

$$\Theta^a = \sum_{\Theta_i \subset F_Y} \vartheta_i \Theta_i, \qquad \Theta^b = \sum_{\Theta_i \not\subset F_Y} \vartheta_i \Theta_i.$$

Similarly, $\bar{\Theta} = \bar{\Theta}^a + \bar{\Theta}^b + \bar{\Theta}^c$, where $\bar{\Theta}^a$ and $\bar{\Theta}^b$ are proper transforms of Θ^a and Θ^b, respectively, and components of $\bar{\Theta}^c = \bar{\Theta} - \bar{\Theta}^a - \bar{\Theta}^b$ are proper transforms of exceptional divisors of μ.

It is clear that $\Theta^b, \bar{\Theta}^b \in \Phi_{\mathbf{sm}}$ and $\bar{\Theta}^c \in (5/6, 1)$. Since the coefficients of Θ^a (as well as $\bar{\Theta}^a$) are of the form

$$\vartheta_i = 1 - 1/m_i + c/m_i \geq c > 5/6 + \epsilon,$$

we have $\Theta^a \in (5/6 + \epsilon, 1)$. By our assumptions $\Theta^a \neq 0$.

We need the following result of Alexeev [2]:

Theorem 4.10. *Fix $\epsilon > 0$. Consider the class of all projective log surfaces (S, Θ) such that $-(K_S + \Theta)$ is nef and $\operatorname{totaldiscr}(S, \Theta) > -1 + \epsilon$ excluding only the case*

- *$\Theta = 0$, $K_S \equiv 0$ and the singularities of S are at worst Du Val.*

Then the class $\{S\}$ is bounded, i.e. S belongs to a finite number of algebraic families.

4.10.1. Let $\bar{\Theta}_1$ be a component of $\bar{\Theta}^a$. Then $\vartheta_1 > 5/6 + \epsilon$. Since $\rho(\bar{S}) = 1$, every two components of $\bar{\Theta}$ intersects each other. Applying Lemma 3.3 we obtain

$$\vartheta_j \leq 11/6 - \vartheta_1 < 11/6 - 5/6 - \epsilon = 1 - \epsilon$$

for all $j \neq 1$. Since $\bar{\Theta}^b \in \Phi_{\mathbf{sm}}$, there is only a finite number of possibilities for the coefficients of $\bar{\Theta}^b$ (and Θ^b).

4.10.2. If $\bar{\Theta}^a$ has at least two components, say $\bar{\Theta}_1$ and $\bar{\Theta}_2$, then by Lemma 3.3 the inequality $\vartheta_k < 1-\epsilon$ holds for all ϑ_k. Thus

$$\operatorname{totaldiscr}(S,\Theta) = \operatorname{totaldiscr}(\bar{S},\bar{\Theta}) > -1+\epsilon.$$

Apply 4.10 to (S,Θ).

For all coefficients of Θ we have $\vartheta_i \ge 1/2$. Fix a very ample divisor H on S. Then $H\cdot\sum\Theta_i \le 2H\cdot K_S \le \mathrm{Const}$. This shows that the pair $(S, \operatorname{Supp}(\Theta))$ is also bounded.

As above, $(S, \operatorname{Supp}(\Theta))$ is bounded. From the equality $0 = K_S^2 + K_S\cdot\Theta^a + K_S\cdot\Theta^b$ we obtain

$$\sum_{\Theta_i\not\subset F_Y} (1-1/m_i + c/m_i)(K_S\cdot\Theta_i) = -K_S^2 - K_S\cdot\Theta^b,$$

where $1-1/m_i+c/m_i < 1-\epsilon$. This gives us a finite number of possibilities for c.

4.10.3. Assume that $\bar{\Theta}^a = \vartheta_1\bar{\Theta}_1$, where $\vartheta_1 = 1-1/m_1+c/m_1$. If $\vartheta_1 < 1-\epsilon$, then we can argue as above. Let $\vartheta_1 \ge 1-\epsilon$. Then Θ_1 is the only divisor with discrepancy $a(\Theta_1,\Theta) \le -1+\epsilon$. Put $\Lambda := \Theta - \vartheta_1\bar{\Theta}_1$. Then $a(\Theta_1,\Lambda)=0$, so $\operatorname{totaldiscr}(S,\Lambda) > -1+\epsilon$. Note that Θ_1 is ample (because $\Theta_1 = (F_Y\big|_S)_{\mathrm{red}}$ and F_Y is f-ample, see 2.6, (iii)). Hence $-(K_S+\Lambda)$ is also ample. By 4.10 $(S,\operatorname{Supp}(\Lambda))$ is bounded and so is $(S,\operatorname{Supp}(\Theta))$. As in 4.10.2, there is only a finite number of possibilities for c.

The following example illustrates our proof:

Example 4.11. Notation as in Example 1.4. Assume that $\gcd(6,r) = 1$. Let $f\colon Y\to X$ be the weighted blowup with weights $(3r,2r,6)$. Then f is an inductive blowup of (X,cF) and the exceptional divisor S is isomorphic to $\mathbb{P}(3r,2r,6)\simeq\mathbb{P}^2$. It is easy to compute that $\Theta = \operatorname{Diff}_S(cF_Y) = \frac{1}{2}L_1 + \frac{2}{3}L_2 + \frac{r-1}{r}L_3 + cL_0$, where $c = 5/6+1/r$ and L_1, L_2, L_3, L_0 are lines on $S\simeq\mathbb{P}^2$ given by equations $x=0$, $y=0$, $z=0$ and $x+y+z=0$, respectively. Thus $\operatorname{discr}(S,\Theta)\ge -5/6$ and $\bar{S} = \widetilde{S} = S \simeq \mathbb{P}^2$.

Concluding remark. (i) Using the same arguments one can see that see that the set $\mathcal{T}_3$ in Theorem 1.2 can be replaced with $\mathcal{T}_3(\Phi_{\mathbf{sm}})$, the set of all values $c(X,\Omega,F)$ with $\Omega\in\Phi_{\mathbf{sm}}$.

(ii) We expect that our proof of Theorem 1.2 can be generalized in higher dimensions modulo the following facts: the log MMP, boundedness result 4.10 and lemmas 3.1 and 3.2. Also we hope that our method allow us to get the complete description of $\mathcal{T}_3\cap[5/6,1]$.

References

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